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On the Evaluation of the Evolution Operator $Z_{\rm Reg}(R_2, R_1)$ in the Diakonov–Petrov Approach to the Wilson Loop

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We evaluate the evolution operator $Z_{\rm Reg}(R_2, R_1)$ introduced by Diakonov and Petrov for the definition of the Wilson loop in terms of a path integral over gauge degrees of freedom. We use the procedure suggested by Diakonov and Petrov (*Physics Letters B* **224** (1989) 131) and show that the evolution operator vanishes.

KEY WORDS: path integral; Wilson loop; gauge theory.

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In Ref. Dyakonov and Petrov (1989) for the representation of the Wilson loop in terms of the path integral over gauge degrees of freedom Diakonov and Petrov used the functional $Z(R_2, R_1)$ defined by (see Eq. (8) of Ref. Dyakonov and Petrov (1989))

$$Z(R_2, R_1) = \int_{R_1}^{R_2} DR(t)\, \exp\left(iT \int_{t_1}^{t_2} {\rm Tr}\,(i R\, \dot{R}\, \tau_3)\right), \tag{1}$$

where $\dot{R} = dR/dt$ and $T = 1/2, 1, 3/2, \ldots$ is the colour isospin quantum number. According to Diakonov and Petrov $Z(R_2, R_1)$ should be regularized by the analogy to an axial–symmetric top. The regularized expression of $Z(R_2, R_1)$ has been determined in Eq. (9) of Ref. Dyakonov and Petrov (1989) and reads

$$Z_{\rm Reg}(R_2, R_1) = \int_{R_1}^{R_2} DR(t)\, \exp\left(i \int_{t_1}^{t_2} \left[\frac{1}{2}\, I_{\perp}\, \left(\Omega_1^2 + \Omega_2^2\right) + \frac{1}{2}\, I_{\parallel}\, \Omega_3^2 + T\, \Omega_3\right]\right), \tag{2}$$

where $\Omega_a = i\, {\rm Tr}(R\, \dot{R}\, \tau_a)$ are angular velocities of the top, τ_a are Pauli matrices $a = 1, 2, 3$, $I_{\perp}$ and $I_{\parallel}$ are the moments of inertia of the top which should be taken

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to zero. According to the prescription of Ref. Dyakonov and Petrov (1989) one should take first the limit $I_{\parallel} \to 0$ and then $I_{\perp} \to 0$. For the confirmation of the result, given in Eq. (13) of Ref. Dyakonov and Petrov (1989),

$$Z_{\mathrm{Reg}}(R_2, R_1) = (2T + 1)\, D^T_{TT}(R_2 R_1^{\dagger}) = (2T + 1)\, D^T_{-T-T}(R_1 R_2^{\dagger}), \tag{3}$$

where $D^T(U)$ is a Wigner rotational matrix in the representation T, Diakonov and Petrov suggested to evaluate the evolution operator (2) explicitly via the discretization of the path integral over R. The discretized form of the path integral Eq. (2) is given by Eq. (14) of Ref. Dyakonov and Petrov (1989) and reads

$$\begin{aligned} Z_{\mathrm{Reg}}(R_{N+1}, R_0) = {} & \lim_{\substack{N \to \infty \\ \delta \to 0}} \mathcal{N} \int \prod_{n=1}^{N} dR_n \\ & \times \exp\left[\sum_{n=0}^{N}\left(-i\,\frac{I_{\perp}}{2\delta}\,\left[(\mathrm{Tr}\, V_n \tau_1)^2 + (\mathrm{Tr}\, V_n \tau_2)^2\right]\right.\right. \\ & \left.\left. -\, i\,\frac{I_{\parallel}}{2\delta}\,(\mathrm{Tr}\, V_n \tau_3)^2 - T\,(\mathrm{Tr}\, V_n \tau_3)\right)\right], \end{aligned} \tag{4}$$

where $R_n = R(s_n)$ with $s_n = t_1 + n\,\delta$ and $V_n = R_n R^{\dagger}_{n+1}$ are the relative orientations of the top at neighbouring points (Dyakonov and Petrov, 1989). The normalization factor $\mathcal{N}$ is determined by

$$\mathcal{N} = \left(\frac{I_{\perp}}{2\pi i \delta}\sqrt{\frac{I_{\parallel}}{2\pi i \delta}}\right)^{N+1}. \tag{5}$$

(see Eq. (19) of Ref. Dyakonov and Petrov (1989)). Following the prescription of Ref. Dyakonov and Petrov (1989) one should take the limits $\delta \to 0$ and $I_{\parallel}, I_{\perp} \to 0$ but keeping the ratios I_i/δ, where $(i = \parallel, \perp)$, much greater than unity, $I_i/\delta \gg 1$.

The main point of the evaluation of the path integral is to show that the evolution operator $Z_{\mathrm{Reg}}(R_2, R_1)$ given by the path integral (2) reduces to the representation in the form of *a sum over possible intermediate states*, i.e. eigenfunctions of the axial–symmetric top (Dyakonov and Petrov, 1989)

$$Z_{\mathrm{Reg}}(R_2, R_1) = \sum_{J=0}^{\infty} \sum_{m=-J}^{J} (2J + 1)\, D^J_{mm}(R_2 R_1^{\dagger})\, e^{-i(t_2 - t_1)\, E_{Jm}}, \tag{6}$$

(see Eq. (12) of Ref. Dyakonov and Petrov (1989)), where E_{Jm} are the eigenvalues of the Hamiltonian of the axial–symmetric top

$$E_{Jm} = \frac{J(J+1) - m^2}{2I_{\perp}} + \frac{(m - T)^2}{2I_{\parallel}} \tag{7}$$

(see Eq. (11) of Ref. Dyakonov and Petrov (1989)).

According to (Dyakonov and Petrov, 1989) the integral has a saddle–point at $V_n \simeq 1$. For the calculation of the integral around the saddle–point Diakonov and Petrov suggested the following procedure. Let us denote the exponent of Eq. (4) as

$$f[V_n] = -i\,\frac{I_\perp}{2\delta}\,\left[(\mathrm{Tr}\,V_n\tau_1)^2 + (\mathrm{Tr}\,V_n\tau_2)^2\right] - i\,\frac{I_\parallel}{2\delta}\,(\mathrm{Tr}\,V_n\tau_3)^2 - T\,(\mathrm{Tr}\,V_n\tau_3) \quad (8)$$

and represent the exponential in the following form

$$e^{f[V_n]} = \sum_{J=0}^{\infty}\sum_{p=-J}^{J}\sum_{q=-J}^{J}(2J+1)\lambda^J_{pq}D^J_{pq}(V_n). \quad (9)$$

The coefficients λ^J_{pq} are given by

$$\lambda^J_{pq} = \int dU_n\, D^J_{qp}(U_n^\dagger)\, e^{f[U_n]}. \quad (10)$$

Substituting Eq. (10) in Eq. (9) we get the identity

$$e^{f[V_n]} = \sum_{J=0}^{\infty}\sum_{p=-J}^{J}\sum_{q=-J}^{J}(2J+1)\,D^J_{pq}(V_n)\int dU_n\, D^J_{qp}(U_n^\dagger)\, e^{f[U_n]}. \quad (11)$$

Let us show that Eq. (11) is the identity. For this aim we have to use the relation

$$\sum_{J=0}^{\infty}\sum_{p=-J}^{J}\sum_{q=-J}^{J}(2J+1)\,D^J_{pq}(V_n)\,D^J_{qp}(U_n^\dagger) = \sum_{J=0}^{\infty}(2J+1)\,\chi_J[V_nU_n^\dagger]. \quad (12)$$

By using Eq. (12) the r.h.s. of Eq. (11) reads

$$\int dU_n\, e^{f[U_n]}\sum_{J=0}^{\infty}(2J+1)\,\chi_J[V_nU_n^\dagger] = \int dU_n\, e^{f[U_n]}\,\delta(V_nU_n^\dagger) = e^{f[V_n]}, \quad (13)$$

where $\delta(V_nU_n^\dagger)$ is a δ–function defined by

$$\sum_{J=0}^{\infty}(2J+1)\,\chi_J[V_nU_n^\dagger] = \delta(V_nU_n^\dagger). \quad (14)$$

The important consequence of these steps is that dU_n as well as dV_n is a standard Haar measure normalized to unity

$$\int dU_n = \int dV_n = 1. \quad (15)$$

Inserting the expansion Eq. (11), integrationg over R_n ($n = 1, 2, \ldots, N$) and using the orthogonality relation for the group elements we arrive at the expression

$$Z_{\rm Reg}(R_{N+1}, R_0) = \lim_{\substack{N\to\infty\\ \delta\to 0}} \sum_{J=0}^{\infty}\sum_{p=-J}^{J}\sum_{q=-J}^{J}(2J+1) \times D^J_{pq}(R_0 R^{\dagger}_{N+1})\left[Z^J_{qp}\right]^{N+1}, \tag{16}$$

where Z^J_{qp} is defined by

$$Z^J_{qp} = \frac{I_\perp}{2\pi i\delta}\sqrt{\frac{I_\parallel}{2\pi i\delta}}\int dU\, D^J_{qp}(U^\dagger)\, e^{f[U]}. \tag{17}$$

Recall that dU is the Haar measure normalized to unity Eq. (15).

For the subsequent evaluation of the integral over U we follow (Dyakonov and Petrov, 1989) and use $U = e^{i\,\frac{1}{2}\,\vec{\omega}\,\cdot\,\vec{\tau}}$ for the fundamental representation and $D^J_{qp}(U^\dagger) = (e^{-\,i\,\vec{\omega}\,\cdot\,\vec{T}})_{qp}$ for $J \neq 1/2$. In the parameterization $U = e^{i\,\frac{1}{2}\,\vec{\omega}\,\cdot\,\vec{\tau}}$ the Haar measure dU reads

$$dU = \frac{d\omega_1 d\omega_2 d\omega_3}{16\pi^2}\left(\frac{2}{\omega}\,\sin\frac{\omega}{2}\right)^2, \tag{18}$$

where $\omega = \sqrt{\omega_1^2 + \omega_2^2 + \omega_3^2}$. According to (Dyakonov and Petrov, 1989) the integral over U calculated in the limit $I_\parallel/\delta,\ I_\perp/\delta \to \infty$ has a saddle point at $U \simeq 1$ [4]. Expanding the integrand around the saddle–point, keeping only quadric terms and neglecting the contribution of the terms coming from the Haar measure, we get

$$Z^J_{qp} = \frac{I_\perp}{2\pi i\delta}\sqrt{\frac{I_\parallel}{2\pi i\delta}}\int_{-\infty}^{\infty} d\omega_1 \int_{-\infty}^{\infty} d\omega_2 \in_{-\infty}^{\infty} d\omega_3\, \exp\left\{i\,\frac{I_\perp}{2\delta}\,\left(\omega_1^2+\omega_2^2\right) + i\,\frac{I_\parallel}{2\delta}\,\omega_3^2\right\}$$
$$\times\left[\delta_{qp} - \frac{1}{2}\left[\omega_1^2\left(T_1^2\right)_{qp} + \omega_2^2\left(T_2^2\right)_{qp}\right] - \frac{1}{2}\,\omega_3^2\,((T_3+T)^2)_{qp}\right]. \tag{19}$$

Integrating over ω_a ($a = 1, 2, 3$) we arrive at the expression

$$Z^J_{qp} = \delta_{qp}\left\{1 - i\,\delta\left[\frac{(J(J+1)-p^2)}{2I_\perp} + \frac{(p+T)^2}{2I_\parallel}\right]\right\}. \tag{20}$$

[4] Below we do not pay attention to the factor $1/16\pi^2$ that has to be included in the normalization factor $\mathcal{N}$ in the form $(16\pi^2)^{N+1}$.

This agrees with the result obtained (Dyakonov and Petrov, 1989). Substituting Eq. (20) in Eq. (16) we obtain the evolution operator $Z_{\rm Reg}(R_0 R^{\dagger}_{N+1})$ defined by

$$Z_{\rm Reg}(R_{N+1}, R_0) = \lim_{N\to\infty} \sum_{J=0}^{\infty} \sum_{p=-J}^{J} (2J+1)\, D^J_{pp}(R_0 R^{\dagger}_{N+1}) \times \left\{1 - i\,\frac{t_2 - t_1}{N+1}\left[\frac{(J(J+1) - p^2)}{2I_{\perp}} + \frac{(p+T)^2}{2I_{\parallel}}\right]\right\}^{N+1}, \tag{21}$$

where we have used the definition of δ: $\delta = (t_2 - t_1)/(N+1)$ (Dyakonov and Petrov, 1989). Taking the limit $N \to \infty$ and replacing $R_0 \to R_1$ and $R^{\dagger}_{\infty} \to R^{\dagger}_2$ we arrive at the expression

$$Z_{\rm Reg}(R_2, R_1) = \sum_{J=0}^{\infty} \sum_{p=-J}^{J} (2J+1)\, D^J_{pp}(R_1 R^{\dagger}_2) \times \exp\left\{-i\,(t_2 - t_1)\left[\frac{(J(J+1) - p^2)}{2I_{\perp}} + \frac{(p+T)^2}{2I_{\parallel}}\right]\right\}. \tag{22}$$

This expression coincides fully with the result obtained in Dyakonov and Petrov (1989) and reproduces the expansion of the evolution operator (6). However, for the calculation of this expression the contribution of the Haar measure has been missed. Expending the Haar measure (18) in the vicinity of a saddle point we get an additional contribution

$$dU = \frac{d\omega_1 d\omega_2 d\omega_3}{16\pi^2}\left(\frac{2}{\omega}\,\sin\frac{\omega}{2}\right)^2 = \frac{d\omega_1 d\omega_2 d\omega_3}{16\pi^2}\left(1 - \frac{1}{12}\left(\omega_1^2 + \omega_2^2 + \omega_3^2\right)\right). \tag{23}$$

This changes the value Z^J_{qp} in Eq. (19) as follows

$$Z^J_{qp} = \delta_{qp}\left\{1 - i\,\delta\,\frac{1}{12}\left(\frac{2}{I_{\perp}} + \frac{1}{I_{\parallel}}\right) - i\,\delta\left[\frac{(J(J+1) - p^2)}{2I_{\perp}} + \frac{(p+T)^2}{2I_{\parallel}}\right]\right\}. \tag{24}$$

In turn, it is not the complete set of contributions of order $O(\delta/I_{\perp})$ and $O(\delta/I_{\parallel})$ to Z^J_{qp}. In order to take into account all of them we have to expand too the exponential $\exp\ f[U]$ keeping the terms of order $\omega_1^4 I_{\perp}/\delta$, $\omega_2^4 I_{\perp}/\delta$, $\omega_3^4 I_{\parallel}/\delta$ and so on. The corresponding expansion of the exponential $\exp\ f[U]$ reads

$$\exp\ f[U] = \exp\left\{i\frac{I_\perp}{2\delta}\left(\omega_1^2+\omega_2^2\right)+i\frac{I_\parallel}{2\delta}\,\omega_3^2\right\} \times\left[1-i\frac{I_\perp}{24\delta}\left(\omega_1^2+\omega_2^2\right)^2-i\frac{I_\perp+I_\parallel}{24\delta}\left(\omega_1^2+\omega_2^2\right)\omega_3^2 - i\frac{I_\parallel}{24\delta}\omega_3^4+\cdots\right], \tag{25}$$

where ellipses denote the terms that have been taken into account in (19). Collecting like terms we obtain a new value for Z^J_{qp}

$$Z^J_{qp} = \delta_{qp}\left\{1+i\,\delta\,\frac{1}{8}\left(\frac{2}{I_\perp}+\frac{1}{I_\parallel}\right)-i\,\delta\left[\frac{(J(J+1)-p^2)}{2I_\perp}+\frac{(p+T)^2}{2I_\parallel}\right]\right\}. \tag{26}$$

This changes the evolution operator as follows

$$Z_{\rm Reg}(R_2,R_1) = \exp\left\{i\,(t_2-t_1)\frac{1}{8}\left(\frac{2}{I_\perp}+\frac{1}{I_\parallel}\right)\right\}\sum_J^\infty\sum_{p=-J}^J(2J+1)\,D^J_{pp}(R_1R_2^\dagger) \times\exp\left\{-i\,(t_2-t_1)\left[\frac{(J(J+1)-p^2)}{2I_\perp}+\frac{(p+T)^2}{2I_\parallel}\right]\right\}. \tag{27}$$

Hence, the evaluation of the path integral (2) with the account for all contributions of order $O(\delta/I_\perp)$ and $O(\delta/I_\parallel)$ around the saddle–point, including the contributions of the Haar measure and the terms of order $\omega_1^4 I_\perp/\delta$, $\omega_2^4 I_\perp/\delta$, $\omega_3^4 I_\parallel/\delta$ and so on, leads to the result that differs fully from the expansion (6) derived from the quantum mechanical consideration of $Z_{\rm Reg}(R_2, R_1)$ in terms of eigenfunctions of the axial–symmetric top. This means that the path integral (2) representing the evolution operator $Z_{\rm Reg}(R_2, R_1)$ has no relation to the axial–symmetric top and predicts a completely different energy spectrum than that given by Eq. (7) for the quantum axial–symmetric top. In the limit $I_\parallel \to 0$ and $I_\perp \to 0$ the evolution operator vanishes by virtue of the strongly oscillating factors. Thus, the only well defined magnitude of the evolution operator is zero. This confirms fully the results obtained in Faber *et al.* (2000) that the evolution operator $Z_{\rm Reg}(R_2, R_1)$ vanishes.

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